

BURKHOLDER-GUNDY-DAVIS INEQUALITY IN MARTINGALE HARDY SPACES WITH VARIABLE EXPONENT

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ABSTRACT. In this paper, the classical Dellacherie's theorem about stochastic process is extended to variable exponent Lebesgue spaces. As its applications, we obtain variable exponent analogues of several famous inequalities in classical martingale theory, including convexity lemma, Burkholder-Gundy-Davis' inequality and Chevalier's inequality. Moreover, we investigate some other equivalent relations between variable exponent martingale Hardy spaces.

1. INTRODUCTION

Due to their important role in elasticity, fluid dynamics, calculus of variations, differential equations and so on, Musielak-Orlicz spaces and their special case, variable exponent Lebesgue spaces have been got more and more attention in modern analysis and functional space theory. In particular, Musielak-Orlicz spaces were studied by Orlicz and Musielak [19]. Hudzik [14] studied some geometry properties of Musielak-Orlicz spaces. Kovacik and Rakosnik [15], Fan and Zhao [11] investigated various properties of variable exponent Lebesgue spaces and Sobolev spaces. Diening [10] and Cruz-Uribe et al [6, 7] proved the boundedness of Hardy-Littlewood maximal operator on variable exponent Lebesgue function spaces $L^{p(x)}(R^n)$ under the conditions that the exponent $p(x)$ satisfies so called log-Hölder continuity and decay restriction. Many other authors studied its applications to harmonic analysis and some other subjects.

As we well known, the situation of martingale spaces is different from function spaces. For example, the log-Hölder continuity of a measurable function on a probability space can not be defined. Moreover, generally speaking, the "good $-\lambda$ " inequality method used in classical martingale theory can not be used in variable exponent case. However, recently, variable exponent martingale spaces have been paid more attention too. Among others, Aoyama [1] proved some inequalities under the condition that the exponent p is Σ_0 -measurable. Nakai and Sadasue [20] pointed out that the Σ_0 -measurability is not necessary for the boundedness of Doob's maximal operator, and proved that the boundedness holds when every σ -algebra is generated by countable atoms.

The aim of this paper is to establish some variable exponent analogues of several famous inequalities in classical martingale theory. By extending Dellacherie's theorem to variable exponent case we obtain convexity Lemma and Burkholder-Gundy-Davis' inequality and Chevalier's inequality for variable exponent martingale Hardy spaces. Then we investigate some equivalent relations between several variable exponent martingale Hardy spaces, specially, we prove that the two predictable martingale spaces $\mathcal{D}_{p(\cdot)}$ and $\mathcal{Q}_{p(\cdot)}$ are equivalent

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and under regular condition the five martingale Hardy spaces $H_{p(\cdot)}^*$, $H_{p(\cdot)}^S$, $H_{p(\cdot)}^s$, $\mathcal{D}_{p(\cdot)}$ and $\mathcal{Q}_{p(\cdot)}$ with variable exponent $1 \leq p^- \leq p^+ < \infty$ are equivalent (for their definitions, see below).

Let (Ω, Σ, μ) be a non-atomic complete probability space, $L^0(\Omega)$ the set of all measurable functions (i.e. r.v.) on Ω , and E the expectation with respect to Σ . We say that $p \in \mathcal{P}$, if $p \in L^0(\Omega)$ with $1 \leq p(\omega) \leq \infty$. For $p \in \mathcal{P}$, denote $\Omega_\infty = \{\omega \in \Omega, p(\omega) = \infty\}$, and define variable exponent Lebesgue space as follows:

$$L^{p(\cdot)} = \{u \in L^0(\Omega) : \exists \gamma > 0, \rho_{p(\cdot)}(\gamma u) < \infty\},$$

where the modular

$$(1.1) \quad \rho_{p(\cdot)}(u) = \int_{\Omega \setminus \Omega_\infty} |u(\omega)|^{p(\omega)} d\mu + \text{ess sup}_{\omega \in \Omega_\infty} |u(\omega)|.$$

For every $u \in L^{p(\cdot)}$, its Luxemburg norm is defined by

$$(1.2) \quad \|u\|_{p(\cdot)} = \inf\{\gamma > 0 : \rho_{p(\cdot)}(\frac{u}{\gamma}) \leq 1\}.$$

We denote by p^+ and p^- the below index and upper index of p , i.e.,

$$p^- = \text{ess inf}_{\omega \in \Omega} p(\omega), \quad p^+ = \text{ess sup}_{\omega \in \Omega} p(\omega),$$

and p 's conjugate index is $p'(\omega)$, i.e., $\frac{1}{p(\omega)} + \frac{1}{p'(\omega)} = 1$.

Here we mention some basic properties of $L^{p(\cdot)}$, the proofs of which are standard and similar to classical function spaces, for example, see [11, 15].

Lemma 1.1. *Let $p \in \mathcal{P}$ with $p^+ < \infty$, then*

- (1) $\rho_{p(\cdot)}(u) < 1$ ($= 1, > 1$) *if and only if* $\|u\|_{p(\cdot)} < 1$ ($= 1, > 1$).
- (2) $\rho_{p(\cdot)}(\frac{u}{\|u\|_{p(\cdot)}}) = 1, \forall u \in L^{p(\cdot)}$ *with* $0 < \|u\|_{p(\cdot)} < \infty$.
- (3) $\rho_{p(\cdot)}(u) \leq \|u\|_{p(\cdot)},$ *if* $\|u\|_{p(\cdot)} \leq 1$.
- (4) $(L^{p(\cdot)}, \|\cdot\|_{p(\cdot)})$ *is a Banach space.*
- (5) *If* $u \in L^{p(\cdot)}, v \in L^{p'(\cdot)},$ *then*

$$(1.3) \quad |Euv| \leq C \|u\|_{p(\cdot)} \|v\|_{p'(\cdot)},$$

where C is a positive constant depending only on p .

- (6) *If* $u_n \in L^{p(\cdot)},$ *then* $\|u_n - u\|_{p(\cdot)} \rightarrow 0$ *if and only if* $\rho_{p(\cdot)}(u_n - u) \rightarrow 0$.

Lemma 1.2. *Let $p \in \mathcal{P}$ and $s > 0$ such that $sp^- \geq 1$, then*

$$(1.4) \quad \| |u|^s \|_{p(\cdot)} = \|u\|_{sp(\cdot)}^s.$$

Lemma 1.3. *Let $p, q \in \mathcal{P}$, then $L^{p(\cdot)} \subset L^{q(\cdot)}$ if and only if $p(\omega) \geq q(\omega)$ a.e., and in this case the embedding is continuous with*

$$(1.5) \quad \|f\|_{q(\cdot)} \leq 2 \|f\|_{p(\cdot)}, \quad \forall f \in L^{p(\cdot)}.$$

Let us fix some notation in martingale theory.

Let $(\Sigma_n)_{n \geq 0}$ be a stochastic basis, i.e., a nondecreasing sequence of sub- σ -algebras of Σ with $\Sigma = \bigvee \Sigma_n$, $f = (f_n)_{n \geq 0}$ a martingale adapted to $(\Sigma_n)_{n \geq 0}$ with its difference sequence $(df_n)_{n \geq 0}$, where $df_n = f_n - f_{n-1}$ (with convention $f_{-1} \equiv 0$ and $\Sigma_{-1} = \{\Omega, \emptyset\}$). We denote by E_n the conditional expectation with respect to Σ_n . For a martingale

$f = (f_n)_{n \geq 0}$, we define its maximal function, square function and conditional square function as usual:

$$f^* = \sup_{n \geq 0} |f_n|, \quad S(f) = \left(\sum_{n=0}^{\infty} |df_n|^2 \right)^{\frac{1}{2}}, \quad s(f) = \left(\sum_{n=0}^{\infty} E_{n-1} |df_n|^2 \right)^{\frac{1}{2}}.$$

For $p \in \mathcal{P}$, the variable exponent martingale Lebesgue space $L^{p(\cdot)}$ and the martingale Hardy spaces $H_{p(\cdot)}^*$, $H_{p(\cdot)}^S$ and $H_{p(\cdot)}^s$ are defined as follows:

$$\begin{aligned} L^{p(\cdot)} &= \{f = (f_n) : \|f\|_{p(\cdot)} = \sup \|f_n\|_{p(\cdot)} < \infty\}, \\ H_{p(\cdot)}^* &= \{f = (f_n) : \|f\|_{H_{p(\cdot)}^*} = \|f^*\|_{p(\cdot)} < \infty\}, \\ H_{p(\cdot)}^S &= \{f = (f_n) : \|f\|_{H_{p(\cdot)}^S} = \|S(f)\|_{p(\cdot)} < \infty\}, \\ H_{p(\cdot)}^s &= \{f = (f_n) : \|f\|_{H_{p(\cdot)}^s} = \|s(f)\|_{p(\cdot)} < \infty\}. \end{aligned}$$

The structure of this paper is as follows. After some preliminaries about variable exponent Lebesgue spaces over a probability space, in section 2 we mainly deal with the extension of Dellacherie's theorem and the convexity lemma to variable exponent case. In section 3 we establish the variable exponent analogues of Burkholder-Gundy-Davis' inequality and Chevalier's inequality. In the last section we first prove the equivalence between two martingale spaces with predictable control, then investigate some equivalent relations between five variable exponent martingale spaces under regular condition.

Through this paper, we always denote by C some positive constant, it may be different in each appearance, and denote by $C_{p(\cdot)}$ a constant depending only on p . Moreover, we say that two norms on X are equivalent, if the identity is continuous in double directions, i.e., there is a constant $C > 0$ such that

$$C^{-1} \|u\|_1 \leq \|u\|_2 \leq C \|u\|_1, \quad \forall u \in X.$$

2. SOME LEMMAS

In this section we prove some lemmas, which will be needed in the sequel.

Lemma 2.1. *Let $p \in \mathcal{P}$ with $1 \leq p(\omega) \leq \infty$, then every martingale or nonnegative submartingale $f = (f_n)$ satisfying $\sup \|f_n\|_{p(\cdot)} < \infty$ converges a.e. to a measurable function $f_\infty \in L^{p(\cdot)}$.*

Proof. Since $p(\omega) \geq 1$, from Lemma 1.3 we have

$$\sup_{n \geq 0} \|f_n\|_1 \leq 2 \sup_{n \geq 0} \|f_n\|_{p(\cdot)} < \infty.$$

By Doob's martingale convergence theorem, $f_n \rightarrow f_\infty$ a.e., by Fatou lemma, $f_\infty \in L^{p(\cdot)}$. \square

In classical theory, Dellacherie exploited a special approach to prove convex Φ -function inequalities for martingales. It was first formulated in [9], also see [18]. The following lemma generalizes Dellacherie theorem to variable exponent case.

Lemma 2.2. *Let $p \in \mathcal{P}$ with $1 \leq p^- \leq p^+ < \infty$, v be a non-negative r.v., $(u_n)_{n \geq 0}$ a nonnegative, nondecreasing adapted sequence satisfying*

$$(2.1) \quad E(u_\infty - u_{n-1} | \Sigma_n) \leq E(v | \Sigma_n), \quad \forall n \geq 0,$$

or a nonnegative, nondecreasing predictable sequence with $u_0 = 0$ and

$$(2.2) \quad E(u_\infty - u_n | \Sigma_n) \leq E(v | \Sigma_n), \quad \forall n \geq 0.$$

then

- (1) $Eu_\infty^p \leq p^+ Evu_\infty^{p-1}$,
- (2) $\rho_{p(\cdot)}(u_\infty) \leq C \rho_{p(\cdot)}(v)$,
- (3) $\|u_\infty\|_{p(\cdot)} \leq C \|v\|_{p(\cdot)}$, if $v \in L^{p(\cdot)}$,

where C is a positive constant depending only on p .

Proof. 1° First we assume that p is Σ_n -measurable for some n , then there is a simple function sequence $\{s_i\}$ such that all s_i are Σ_n -measurable, $s_i \geq 1$ and $s_i \uparrow p$.

Let $s_i = \sum_{j=1}^{J_i} a_{i,j} \chi_{A_{i,j}}$, where $A_{i,j} \in \Sigma_n$ with $A_{i,j} \cap A_{i,j'} = \emptyset$, $j \neq j'$, $\bigcup_{j=1}^{J_i} A_{i,j} = \Omega$. For each $A_{i,j}$, take $\Phi(t) = t^{a_{i,j}}$, replace $(u_n)_{n \geq 1}$ and v by $(u'_m)_{m \geq 0}$ and v' , respectively. Here, $u'_m = u_{m+n} \chi_{A_{i,j}}$, $\Sigma'_m = \Sigma_{m+n}$, $v' = v \chi_{A_{i,j}}$. In view of the Σ_n -measurability of $A_{i,j}$, inequality (2.1) becomes

$$\begin{aligned}
 E(u'_\infty - u'_{m-1} | \Sigma'_m) &= E(u_\infty \chi_{A_{i,j}} - u_{m+n-1} \chi_{A_{i,j}} | \Sigma_{m+n}) \\
 &= E(u_\infty - u_{m+n-1} | \Sigma_{m+n}) \chi_{A_{i,j}} \\
 &\leq E(v | \Sigma_{m+n}) \chi_{A_{i,j}} \\
 &= E(v \chi_{A_{i,j}} | \Sigma_{m+n}) = E(v' | \Sigma'_m), \quad \forall m \geq 0.
 \end{aligned}
 \tag{2.3}$$

Similarly, (2.2) becomes

$$E(u'_\infty - u'_m | \Sigma'_m) \leq E(v' | \Sigma'_m), \quad \forall m \geq 0. \tag{2.4}$$

By classical Dellacherie theorem, we get

$$Eu_\infty^{a_{i,j}} \chi_{A_{i,j}} \leq Ea_{i,j} v u_\infty^{a_{i,j}-1} \chi_{A_{i,j}},$$

and

$$Eu_\infty^{s_i} = \sum_{j=1}^{J_i} Eu_\infty^{a_{i,j}} \chi_{A_{i,j}} \leq \sum_{j=1}^{J_i} a_{i,j} Evu_\infty^{a_{i,j}-1} \chi_{A_{i,j}} \leq p^+ Evu_\infty^{s_i-1}. \tag{2.5}$$

If $Evu_\infty^{p-1} < \infty$, it is clear that

$$vu_\infty^{s_i-1} = v |u_\infty \chi_{\{u_\infty < 1\}}|^{s_i-1} + v |u_\infty \chi_{\{u_\infty \geq 1\}}|^{s_i-1},$$

Lebesgue dominated convergence theorem and Levi monotonic convergence theorem give $Evu_\infty^{s_i-1} \rightarrow Evu_\infty^{p-1}$, as $i \rightarrow \infty$. Similarly, from

$$|u_\infty|^{s_i} = |u_\infty \chi_{\{u_\infty < 1\}}|^{s_i} + |u_\infty \chi_{\{u_\infty \geq 1\}}|^{s_i},$$

we get $Eu_\infty^{s_i} \rightarrow Eu_\infty^p$, as $i \rightarrow \infty$. By taking limit on both sides of (2.5), we obtain $Eu_\infty^p \leq p^+ Evu_\infty^{p-1}$, this is (1).

Now suppose that $p \in \mathcal{P}$ only, we claim that there is a sequence $\{p_k\}$ of simple functions such that p_k is Σ_{n_k} -measurable for some n_k and $p_k \uparrow p$, $n_k \uparrow \infty$ as $k \uparrow \infty$. Indeed, we first take a simple function sequence $\{g_k\}$, which is Σ -measurable such that $g_k \uparrow p$. Due to $\Sigma = \sigma(\bigcup_{n=1}^\infty \Sigma_n)$, for every $A \in \Sigma$, there is a sequence $A_k \in \bigcup_{n=1}^\infty \Sigma_n$ such that $\mu(A \triangle A_k) \rightarrow 0$. Since (Σ_n) is increasing, for every g_k , there is a simple function g'_k such that g'_k is Σ_{n_k} -measurable, $g'_k \leq p$ and $\mu\{g_k \neq g'_k\} < 2^{-k}$, so we have $g'_k \rightarrow p$, a.e. Let $p_k = g'_1 \vee \dots \vee g'_k$, then $p_k \uparrow p$, a.e. From previous proof, (1) holds for every p_k . We then obtain (1) for the general case by taking limit.

2° Notice that for $p, q \in \mathcal{P}$ with $\frac{1}{p(\omega)} + \frac{1}{q(\omega)} = 1$, by Young inequality we have

$$p^+ b a^{p-1} \leq \frac{(p^+ b)^p}{p} + \frac{a^p}{q}, \quad a^p = \frac{a^p}{p} + \frac{a^p}{q}, \quad \forall a, b > 0.$$

Letting $a = u_\infty(\omega)$, $b = v(\omega)$ in these inequalities and taking integrals on both sides, then (2) immediately follows from (1).

3° To prove (3), we may assume $\|u_\infty\|_{p(\cdot)} = 1$. Since $\rho_{p'(\cdot)}(u_\infty^{p-1}) = \rho_{p(\cdot)}(u_\infty) = 1$, Lemma 1.1(5) and the proof above of (1) show that

$$1 = \rho_{p(\cdot)}(u_\infty) = Eu_\infty^p \leq p^+ Evu_\infty^{p-1} \leq C\|v\|_{p(\cdot)}\|u_\infty^{p-1}\|_{p'(\cdot)} = C\|v\|_{p(\cdot)},$$

so $\|u_\infty\|_{p(\cdot)} \leq C\|v\|_{p(\cdot)}$. The proof is complete. \square

The following lemma is so called convexity lemma whose classical version belongs to Burkholder, Davis and Gundy, see [3].

Lemma 2.3. *Let $p \in \mathcal{P}$ with $p^+ < \infty$, $(\xi_n)_{n \geq 0}$ be a non-negative r.v. sequence, then there is a constant $C = C_{p(\cdot)} > 0$ such that*

$$(2.6) \quad \left\| \sum_{n=1}^{\infty} E_n \xi_n \right\|_{p(\cdot)} \leq C \left\| \sum_{n=1}^{\infty} \xi_n \right\|_{p(\cdot)}.$$

Proof. Let $v_n = \sum_{k=1}^n \xi_k$, $u_n = \sum_{k=1}^n E_k \xi_k$, for any n we have

$$E_n(u_\infty - u_{n-1}) = E_n(v_\infty - v_{n-1}) \leq E_n v_\infty,$$

so (2.6) follows from Lemma 2.2(3). \square

Lemma 2.4. *Let $p \in \mathcal{P}$ with $2 \leq p \leq p^+ < \infty$, then there is a $C = C_{p(\cdot)}$ such that for every martingale $f = (f_n)$,*

$$(2.7) \quad \|s(f)\|_{p(\cdot)} \leq C \|S(f)\|_{p(\cdot)}.$$

Proof. Let $v_n = S_n(f)^2 = \sum_{k=1}^n |df_k|^2$, $u_n = s_n(f)^2 = \sum_{k=1}^n E_{k-1} |df_k|^2$, then by Lemmas 1.2 and 2.3 we obtain

$$\|s(f)\|_{p(\cdot)} = \|s(f)^2\|_{\frac{p(\cdot)}{2}}^{\frac{1}{2}} \leq C \|S(f)^2\|_{\frac{p(\cdot)}{2}}^{\frac{1}{2}} = C \|S(f)\|_{p(\cdot)}.$$

This is desired. \square

3. BURKHOLDER-GUNDY-DAVIS INEQUALITY

Let us first extend classical Burkholder-Gundy-Davis inequality to variable exponent case, which is one of the most fundamental theorems in martingale theory, see [3].

Theorem 3.1. *Let $p \in \mathcal{P}$ with $1 \leq p^- \leq p^+ < \infty$, then there is a $C = C_{p(\cdot)}$ such that for every martingale $f = (f_n)$,*

$$(3.1) \quad C^{-1} \|f^*\|_{p(\cdot)} \leq \|S(f)\|_{p(\cdot)} \leq C \|f^*\|_{p(\cdot)}.$$

Proof. Here we use Davis' method. For a martingale $f = (f_n)$, we define $g = (g_m)$, $g_m = f_{m+n} - f_{n-1}$, $\Sigma'_m = \Sigma_{m+n}$, $m \geq 0$. Due to the classical Burkholder-Gundy-Davis inequality (in conditioned version), we have

$$(3.2) \quad \begin{aligned} E(f^* - f_{n-1}^* | \Sigma_n) &\leq E(\sup_{m \geq 0} |f_{n+m} - f_{n-1}| | \Sigma_n) \\ &= E(g^* | \Sigma'_0) \leq CE(S(g) | \Sigma'_0) \\ &= CE((S(f)^2 - S_{n-1}(f)^2)^{\frac{1}{2}} | \Sigma_n) \\ &\leq CE(S(f) | \Sigma_n), \end{aligned}$$

and

$$(3.3) \quad \begin{aligned} E(S(f) - S_{n-1}(f)|\Sigma_n) &\leq E((S(f)^2 - S_{n-1}(f)^2)^{\frac{1}{2}}|\Sigma_n) \\ &\leq E(S(g)|\Sigma'_0) \leq CE(g^*|\Sigma'_0) \leq CE(f^*|\Sigma_n). \end{aligned}$$

Using Lemma 2.2, the inequality (3.1) follows from (3.2) and (3.3). \square

Now we prove a more shaper inequality: Chevalier's inequality. For a martingale $f = (f_n)$, consider the functions $M(f)$ and $m(f)$:

$$M(f) = \sup_{n \geq 0} M_n(f), \quad m(f) = \sup_{n \geq 0} m_n(f),$$

where

$$M_n(f) = f_n^* \vee S_n(f), \quad m_n(f) = f_n^* \wedge S_n(f).$$

Theorem 3.2. *Let $p \in \mathcal{P}$ with $1 \leq p^- \leq p^+ < \infty$, then there is a $C = C_{p(\cdot)}$ such that for every martingale $f = (f_n)$,*

$$(3.4) \quad \|M(f)\|_{p(\cdot)} \leq C\|m(f)\|_{p(\cdot)}.$$

Proof. We begin with a well known result (see Long [18], Theorem 3.5.5.): let $g = (g_m)$ be as in the proof of Theorem 3.1 and (D_n) a predictable control of difference sequence (df_n) , i.e. (D_n) is an increasing adapted r.v. sequence with $|df_n| \leq D_{n-1}, n \geq 0$, then there is a $C > 0$ such that

$$(3.5) \quad E_0(M(f)) \leq CE_0(m(f) + D_\infty).$$

Now for any fixed n , we have

$$\begin{aligned} M(f) - M_{n-1}(f) &\leq (f^* - f_{n-1}^*) \vee (S(f) - S_{n-1}(f)) \\ &\leq g^* \vee S(g) = M(g), \end{aligned}$$

and

$$m(g) = g^* \wedge S(g) \leq 2f^* \wedge S(f) \leq 2m(f).$$

Using (3.5) we get

$$\begin{aligned} E(M(f) - M_{n-1}(f)|\Sigma_n) &\leq E(M(g)|\Sigma'_0) \\ &\leq CE(m(g) + D'_\infty|\Sigma'_0) \leq CE(m(f) + D_\infty|\Sigma_n). \end{aligned}$$

Where D' is a predictable control of g with $D' = (D'_m), D'_m = D_{m+n}, D'_\infty = D_\infty$. Lemma 2.2 guarantees that the following inequality holds:

$$(3.6) \quad \|M(f)\|_{p(\cdot)} \leq C\|m(f) + D_\infty\|_{p(\cdot)}.$$

Making f 's Davis decomposition $f = g + h$, with $|dg_n| \leq 4d_{n-1}^*$ and

$$\sum |dh_n| \leq 2d^* + 2 \sum E_{n-1}(d_n^* - d_{n-1}^*),$$

where $d_n^* = \sup_{0 \leq k \leq n} |df_k|$, we have the following estimate:

$$d^* \leq 2f^* \wedge S(f) \leq 2m(f), \quad h^* \vee S(h) \leq \sum |dh_n|$$

and

$$\begin{aligned} m(g) &\leq (f^* + h^*) \wedge (S(f) + S(h)) \\ &\leq f^* \wedge S(f) + \sum |dh_n| = m(f) + \sum |dh_n|. \end{aligned}$$

From Lemma 2.3, we then have

$$\|\sum |dh_n|\|_{p(\cdot)} \leq C\|d^*\|_{p(\cdot)} + C\|\sum E_{n-1}(d_n^* - d_{n-1}^*)\|_{p(\cdot)} \leq C\|d^*\|_{p(\cdot)}.$$

By using (3.6), we obtain

$$\begin{aligned}\|M(f)\|_{p(\cdot)} &\leq \|M(g)\|_{p(\cdot)} + \|M(h)\|_{p(\cdot)} \\ &\leq C\|m(g)\|_{p(\cdot)} + C\|d^*\|_{p(\cdot)} + C\|m(f)\|_{p(\cdot)} \\ &\leq C\|m(f)\|_{p(\cdot)}.\end{aligned}$$

This completes the proof. \square

4. SOME EQUIVALENT RELATIONS BETWEEN MARTINGALE SPACES

Let $p \in \mathcal{P}$ with $1 \leq p \leq p^+ < \infty$, $\lambda = (\lambda_n)$ be a nonnegative and increasing adapted sequence with $\lambda_\infty = \lim_{n \rightarrow \infty} \lambda_n \in L^{p(\cdot)}$. We denote by Λ the set of all such sequences and define two martingale spaces as follows:

$$\mathcal{Q}_{p(\cdot)} = \{f = (f_n) : \exists \lambda \in \Lambda, S_n(f) \leq \lambda_{n-1}, \|f\|_{\mathcal{Q}_{p(\cdot)}} = \inf_{\lambda \in \Lambda} \|\lambda_\infty\|_{p(\cdot)} < \infty\},$$

$$\mathcal{D}_{p(\cdot)} = \{f = (f_n) : \exists \lambda \in \Lambda, |f_n| \leq \lambda_{n-1}, \|f\|_{\mathcal{D}_{p(\cdot)}} = \inf_{\lambda \in \Lambda} \|\lambda_\infty\|_{p(\cdot)} < \infty\}.$$

It is easy to check that both two martingale spaces above are Banach spaces, and as in the classical case the norms of $\mathcal{Q}_{p(\cdot)}, \mathcal{D}_{p(\cdot)}$ can be reached, we call such λ an optimal predictable control of f . We also introduce the following martingale space $\mathcal{A}_{p(\cdot)}$:

$$\mathcal{A}_{p(\cdot)} = \{f = (f_n) : \|f\|_{\mathcal{A}_{p(\cdot)}} = \left\| \sum_{n=0}^{\infty} |df_n| \right\|_{p(\cdot)} < \infty\}.$$

To prove the equivalence between $\mathcal{Q}_{p(\cdot)}$ and $\mathcal{D}_{p(\cdot)}$, we first need the following theorem.

Theorem 4.1. *Let $p \in \mathcal{P}$ with $1 \leq p^- \leq p^+ < \infty$, then there are $C = C_{p(\cdot)}$ such that the following inequalities hold for every martingale $f = (f_n)$,*

$$(4.1) \quad \|f\|_{H_{p(\cdot)}^*} \leq \|f\|_{\mathcal{D}_{p(\cdot)}}, \quad \|f\|_{H_{p(\cdot)}^S} \leq \|f\|_{\mathcal{Q}_{p(\cdot)}},$$

$$(4.2) \quad \|f\|_{H_{p(\cdot)}^*} \leq C\|f\|_{\mathcal{Q}_{p(\cdot)}}, \quad \|f\|_{H_{p(\cdot)}^S} \leq C\|f\|_{\mathcal{D}_{p(\cdot)}}.$$

Proof. The two inequalities in (4.1) are obvious from their definitions. And the two inequalities in (4.2) come from (3.1) and (4.1). \square

We now prove a Davis' decomposition theorem for martingales in $H_{p(\cdot)}^S$ and $H_{p(\cdot)}^*$.

Theorem 4.2. *Let $p \in \mathcal{P}$ with $1 \leq p^- \leq p^+ < \infty$, then*

- (1) *Every $f = (f_n) \in H_{p(\cdot)}^S$ has a decomposition $f = g + h$ with $g \in \mathcal{Q}_{p(\cdot)}, h \in \mathcal{A}_{p(\cdot)}$ such that*

$$(4.3) \quad \|g\|_{\mathcal{Q}_{p(\cdot)}} \leq C\|f\|_{H_{p(\cdot)}^S}, \quad \|h\|_{\mathcal{A}_{p(\cdot)}} \leq C\|f\|_{H_{p(\cdot)}^S}.$$

- (2) *Every $f = (f_n) \in H_{p(\cdot)}^*$ has a decomposition $f = g + h$ with $g \in \mathcal{D}_{p(\cdot)}, h \in \mathcal{A}_{p(\cdot)}$ such that*

$$(4.4) \quad \|g\|_{\mathcal{D}_{p(\cdot)}} \leq C\|f\|_{H_{p(\cdot)}^*}, \quad \|h\|_{\mathcal{A}_{p(\cdot)}} \leq C\|f\|_{H_{p(\cdot)}^*}.$$

Proof. Here we only prove (4.3), the proof of (4.4) is similar.

Let $\lambda = (\lambda_n)$ be an increasing control of $(S_n(f))_{n \geq 0} : |S_n(f)| \leq \lambda_n, \lambda_\infty \in L^{p(\cdot)}$. Define

$$\begin{aligned}dh_n &= df_n \chi_{\{\lambda_n > 2\lambda_{n-1}\}} - E_{n-1}(df_n \chi_{\{\lambda_n > 2\lambda_{n-1}\}}), \\ dg_n &= df_n \chi_{\{\lambda_n \leq 2\lambda_{n-1}\}} - E_{n-1}(df_n \chi_{\{\lambda_n \leq 2\lambda_{n-1}\}})\end{aligned}$$

and $h_n = \sum_{k=0}^n dh_k$, $g_n = \sum_{k=0}^n dg_k$. It is clear that $(h_n)_{n \geq 0}$ and $(g_n)_{n \geq 0}$ are martingales and $f_n = g_n + h_n$, $\forall n \geq 0$. As usual, we have

$$|df_k| \chi_{\{\lambda_k > 2\lambda_{k-1}\}} \leq \lambda_k \chi_{\{\lambda_k > 2\lambda_{k-1}\}} \leq 2\lambda_k - 2\lambda_{k-1},$$

thus

$$\sum_{k=0}^{\infty} |dh_k| \leq 2\lambda_{\infty} + 2 \sum_{k=0}^{\infty} E_{k-1}(\lambda_k - \lambda_{k-1}).$$

Lemma 2.3 guarantees

$$(4.5) \quad \|h\|_{\mathcal{A}_{p(\cdot)}} = \left\| \sum_{k=0}^{\infty} |dh_k| \right\|_{p(\cdot)} \leq C \|\lambda_{\infty}\|_{p(\cdot)}.$$

On the other hand, since $|df_k| \chi_{\{\lambda_k \leq 2\lambda_{k-1}\}} \leq 2\lambda_{k-1}$ and $|dg_k| \leq 4\lambda_{k-1}$, thus

$$\begin{aligned} S_n(g) &\leq S_{n-1}(g) + |dg_n| \leq S_{n-1}(f) + S_{n-1}(h) + 4\lambda_{n-1} \\ &\leq \lambda_{n-1} + 2\lambda_{n-1} + 2 \sum_{k=0}^{n-1} E_{k-1}(\lambda_k - \lambda_{k-1}) + 4\lambda_{n-1}, \end{aligned}$$

so $g \in \mathcal{Q}_{p(\cdot)}$, also due to Lemma 2.3,

$$(4.6) \quad \|g\|_{\mathcal{Q}_{p(\cdot)}} \leq \|7\lambda_{\infty} + 2 \sum_{k=0}^{\infty} E_{k-1}(\lambda_k - \lambda_{k-1})\|_{p(\cdot)} \leq C \|\lambda_{\infty}\|_{p(\cdot)}.$$

Taking $\lambda_n = S_n(f)$, then (4.3) follows from (4.5) and (4.6). \square

The following statement is about the equivalence between $\mathcal{Q}_{p(\cdot)}$ and $\mathcal{D}_{p(\cdot)}$. Refer to Chao and Long [4], also see [22] for the classical version.

Theorem 4.3. *Let $p \in \mathcal{P}$ with $1 \leq p^- \leq p^+ < \infty$, then there is a $C = C_{p(\cdot)} > 0$ such that for every martingale $f = (f_n)$,*

$$(4.7) \quad C^{-1} \|f\|_{\mathcal{D}_{p(\cdot)}} \leq \|f\|_{\mathcal{Q}_{p(\cdot)}} \leq C \|f\|_{\mathcal{D}_{p(\cdot)}}.$$

Proof. Let $f = (f_n) \in \mathcal{D}_{p(\cdot)}$ and $\lambda = (\lambda_n)$ be its optimal predictable control: $|f_n| \leq \lambda_{n-1}$, $\|f\|_{\mathcal{D}_{p(\cdot)}} = \|\lambda_{\infty}\|_{p(\cdot)}$. Since

$$S_n(f) \leq S_{n-1}(f) + |df_n| \leq S_{n-1}(f) + 2\lambda_{n-1},$$

namely, $(S_{n-1}(f) + 2\lambda_{n-1})_{n \geq 0}$ is a predictable control of $(S_n(f))_{n \geq 0}$, then $f \in \mathcal{Q}_{p(\cdot)}$ and it follows from (4.2) that

$$\|f\|_{\mathcal{Q}_{p(\cdot)}} \leq \|S(f) + 2\lambda_{\infty}\|_{p(\cdot)} \leq \|f\|_{H_{p(\cdot)}^S} + 2\|\lambda_{\infty}\|_{p(\cdot)} \leq C \|f\|_{\mathcal{D}_{p(\cdot)}}.$$

Conversely, if $f = (f_n) \in \mathcal{Q}_{p(\cdot)}$ and $\lambda = (\lambda_n)$ is its predictable control. Since

$$|f_n| \leq |f_{n-1}| + |df_n| \leq f_{n-1}^* + 2\lambda_{n-1},$$

then $f \in \mathcal{D}_{p(\cdot)}$. Using (4.2) again we obtain

$$\|f\|_{\mathcal{D}_{p(\cdot)}} \leq \|f^* + 2\lambda_{\infty}\|_{p(\cdot)} \leq \|f\|_{H_{p(\cdot)}^*} + 2\|\lambda_{\infty}\|_{p(\cdot)} \leq C \|f\|_{\mathcal{Q}_{p(\cdot)}}.$$

The proof is complete. \square

Theorem 4.4. *If $p \in \mathcal{P}$ with $1 \leq p^- \leq p^+ < \infty$, then there is a $C = C_{p(\cdot)}$ such that for every martingale $f = (f_n)$ with $f_0 = 0$,*

$$(4.8) \quad \|f\|_{H_{p(\cdot)}^s} \leq C \|f\|_{\mathcal{D}_{p(\cdot)}}, \quad \|f\|_{H_{p(\cdot)}^S} \leq C \|f\|_{\mathcal{D}_{p(\cdot)}}.$$

Proof. Here we use Garsia's idea which was used to prove theorem 4.1.2 in [12]. Let $f \in \mathcal{D}_{p(\cdot)}$ and $\lambda = (\lambda_n)$ be its optimal predictable control: λ_n is positive and increasing, $|f_n| \leq \lambda_{n-1}$ and $\|\lambda_\infty\|_{p(\cdot)} = \|f\|_{\mathcal{D}_{p(\cdot)}}$. Define

$$g_n = \sum_{i=1}^n \frac{df_i}{\sqrt{\lambda_{i-1}}}, \quad \forall n \geq 1.$$

A simple computation shows that

$$g_n = \sum_{i=1}^n \frac{f_i - f_{i-1}}{\sqrt{\lambda_{i-1}}} = \frac{f_n}{\sqrt{\lambda_{n-1}}} + \sum_{i=1}^{n-1} \frac{f_i}{\sqrt{\lambda_{i-1}\lambda_i}} (\sqrt{\lambda_i} - \sqrt{\lambda_{i-1}}),$$

and

$$(4.9) \quad |g_n| \leq 2\sqrt{\lambda_{n-1}}, \quad g^* \leq 2\sqrt{\lambda_\infty}, \quad Eg^{*2} \leq 4E\lambda_\infty < \infty.$$

So $g = (g_n)$ is an L^2 -bounded martingale, it converges to $g_\infty = \sum_{i=1}^\infty \frac{df_i}{\sqrt{\lambda_{i-1}}}$ a.e. and in L^2 . Notice that

$$f_n = \sum_{i=1}^n \sqrt{\lambda_{i-1}} dg_i, \quad \forall n \geq 1,$$

then

$$(4.10) \quad s_n^2(f) = \sum_{i=1}^n \lambda_{i-1} E_{i-1} |dg_i|^2 \leq \lambda_{n-1} s_n^2(g).$$

and

$$(4.11) \quad S_n^2(f) \leq \lambda_{n-1} S_n^2(g).$$

By Hölder inequality we have

$$(4.12) \quad Es(f)^p \leq E\lambda_\infty^{\frac{p}{2}} s(g)^p \leq (E\lambda_\infty^p)^{\frac{1}{2}} (Es(g)^{2p})^{\frac{1}{2}}.$$

Using Lemmas 2.2(2), 2.4 and inequalities (3.3), (4.9) we obtain

$$\begin{aligned} Es(f)^p &\leq C(E\lambda_\infty^p)^{\frac{1}{2}} (Es(g)^{2p})^{\frac{1}{2}} \\ &\leq C(E\lambda_\infty^p)^{\frac{1}{2}} (Eg^{*2p})^{\frac{1}{2}} \\ &\leq C(E\lambda_\infty^p)^{\frac{1}{2}} (E\lambda_\infty^p)^{\frac{1}{2}} = C(E\lambda_\infty^p), \end{aligned}$$

this implies the first inequality of (4.8).

A similar argument gives the second inequality of (4.8). \square

Now we consider some equivalent relations between five martingale spaces under regular condition. In [22], Weisz called a martingale $f = (f_n)$ is previsible, if there is a real number $R > 0$ such that

$$(4.13) \quad |df_n| \leq RE_{n-1} |df_n|, \quad \forall n \geq 0,$$

and proved that if it holds for all martingale with the same constant R , then the stochastic basis (Σ_n) is regular (refer to Garsia [12] for its definition). He also proved if (Σ_n) is regular, then all the spaces H_p^* , H_p^S , H_p^s , \mathcal{D}_p and \mathcal{Q}_p are equivalent for $0 < p < \infty$. The following theorem is the variable exponent analogues.

Theorem 4.5. *If $p \in \mathcal{P}$ with $1 \leq p^- \leq p^+ < \infty$ and the stochastic basis (Σ_n) is regular, then the martingale spaces $H_{p(\cdot)}^*$, $H_{p(\cdot)}^S$, $H_{p(\cdot)}^s$, $\mathcal{D}_{p(\cdot)}$ and $\mathcal{Q}_{p(\cdot)}$ are equivalent.*

Proof. Under the regular condition, we notice that

$$S_n(f) \leq S_{n-1}(f) + |df_n| \leq S_{n-1}(f) + RE_{n-1}|df_n| \leq S_{n-1}(f) + RE_{n-1}S_n(f),$$

that is, $(S_{n-1}(f) + RE_{n-1}S_n(f))_{n \geq 0}$ is a predictable control of $(S_n(f))_{n \geq 0}$. Since

$$E_{n-1}S_n(f) \leq S_{n-1}(f) + E_{n-1}(S_n(f) - S_{n-1}(f)),$$

Lemma 2.3 gives

$$\|f\|_{\mathcal{Q}_{p(\cdot)}} \leq 2\|S(f)\|_{p(\cdot)} + R\left\|\sum_{n=0}^{\infty} E_{n-1}(S_n(f) - S_{n-1}(f))\right\|_{p(\cdot)} \leq C\|S(f)\|_{p(\cdot)},$$

where C is some constant depending only on p and R . Then it follows from Theorems 3.1 and 4.3 that

$$(4.14) \quad \|f\|_{H_{p(\cdot)}^S} \leq C\|f\|_{H_{p(\cdot)}^*} \leq C\|f\|_{\mathcal{D}_{p(\cdot)}} \leq C\|f\|_{\mathcal{Q}_{p(\cdot)}} \leq C\|f\|_{H_{p(\cdot)}^S}.$$

It remains to prove

$$(4.15) \quad C^{-1}\|f\|_{H_{p(\cdot)}^S} \leq \|f\|_{H_{p(\cdot)}^*} \leq C\|f\|_{H_{p(\cdot)}^S}.$$

The first inequality comes from Theorem 4.4 and (4.13). Due to the regularity, we have $S_n(f) \leq Rs_n(f)$ and $S(f) \leq Rs(f)$, so the second inequality follows directly. The proof is complete. \square

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